Block Eigenvalues of Block Compound Matrices

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ABSTRACT

We obtain some results about the block eigenvalues of block compound matrices and additive block compound matrices. Assuming that a certain block Vandermonde matrix is nonsingular, we generalize known results for (scalar) compound and additive compound matrices.

1. THE BLOCK-EIGENVALUE PROBLEM

This paper concerns block eigenvalues of matrices. Our interest was attracted to block-eigenvalue questions by a technical report by Dennis, Traub, and Weber [1].

In what follows we consider only complex matrices partitioned into commuting blocks.

Let us introduce some notation: $P_n(\mathbb{C})$ is a set of s-order square complex matrices that commute in pairs; $M_{p,q}(P_n)$ is the set of matrices partitioned into $p \times q$ blocks each belonging to $P_n(\mathbb{C})$; and $M_r(P_n)$ is the case $p = q = r$. The matrices in $M_{p,q}(P_n)$ are complex matrices of type $pn \times qn$; the matrices in $M_r(P_n)$ are rn-order square complex matrices.

Then given a matrix $A \in M_m(P_n)$, we consider matrices $\Lambda \in P_n(\mathbb{C})$ such that there exists a matrix $X \in M_{m,1}(P_n)$, of full rank, satisfying

$$AX = X\Lambda. \tag{1.1}$$

Λ is called a *block eigenvalue* of the matrix A, and X, of full rank, is the corresponding *block eigenvector* of the matrix A.

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One important result is that *the eigenvalues of* Λ *are eigenvalues of* A [1].

A set of block eigenvalues of a matrix is a *complete set* if the set of all the eigenvalues of these block eigenvalues is the set of eigenvalues of the matrix [1].

By the *formal determinant* of a matrix $F \in M_r(P_n)$ we mean the matrix $\det F$ which we obtain by developing the determinant of F, considering the (commuting) blocks as elements. By $\operatorname{Det} G$ we denote the determinant of a matrix G. It is known [4] that

$$\operatorname{Det} F = \operatorname{Det}(\det F). \tag{1.2}$$

2. BLOCK SIMILARITY

In this section we look for conditions for a matrix partitioned into commuting blocks to be similar to a block-diagonal matrix. We need the following

DEFINITION 2.1. The block vectors $V_1, V_2, \dots, V_k \in M_{m,1}(P_n)$ are block-linearly independent if, with $A_i \in P_n(\mathbb{C})$ $(i = 1, 2, \dots, k)$,

$$\sum_{i=1}^{k} V_i A_i = 0 \in M_{m,1}(P_n) \quad \text{implies} \quad A_i = 0 \in P_n(\mathbb{C}) \quad (i = 1, 2, \dots, k).$$

LEMMA 2.1. *The block vectors* $V_1, V_2, \dots, V_m \in M_{m,1}(P_n)$ *are block-linearly independent if and only if the matrix* $(V_1 \ V_2 \ \cdots \ V_m) \in M_m(P_n)$ *is nonsingular.*

Proof. We have

$$\sum_{i=1}^{m} V_i A_i = 0 \in M_{m,1}(P_n) \quad \Leftrightarrow \quad (V_1 \ \cdots \ V_m) \begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

And we have $A_i = 0 \in P_n(\mathbb{C})$ $(i = 1, 2, \dots, m)$ if and only if the matrix $(V_1 \ \cdots \ V_m)^{-1}$ exists. ∎

Now we are able to state:

PROPOSITION 2.1. *Let* $\{\Lambda_1, \Lambda_2, \dots, \Lambda_m\}$ *be a set of block eigenvalues* $\Lambda_i \in P_n(\mathbb{C})$ *of the matrix* $A \in M_m(P_n)$, *such that the* (block-Vandermonde) *matrix*

$$V(\Lambda_1 \Lambda_2 \cdots \Lambda_m) := \begin{pmatrix} I_n & I_n & \cdots & I_n \\ \Lambda_1 & \Lambda_2 & \cdots & \Lambda_m \\ \Lambda_1^2 & \Lambda_2^2 & \cdots & \Lambda_m^2 \\ \cdots & \cdots & \cdots & \cdots \\ \Lambda_1^{m-1} & \Lambda_2^{m-1} & \cdots & \Lambda_m^{m-1} \end{pmatrix} \in M_m(P_n)$$

is nonsingular. Then the block eigenvectors $X_i \in M_{m,1}(P_n)$ *corresponding to* $\Lambda_i \in P_n(\mathbb{C})$ $(i = 1, 2, \dots, m)$ *are block-linearly independent.*

Proof. Let us define the unit matrix $I \in M_m(P_n)$ to be

$$I = \begin{pmatrix} I_n & 0 & \cdots & 0 \\ 0 & I_n & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & I_n \end{pmatrix},$$

where I_n is the unit matrix in $P_n(\mathbb{C})$, and the scalar block product $\Lambda_k I$ by

$$\Lambda_k I = \begin{pmatrix} \Lambda_k & 0 & \cdots & 0 \\ 0 & \Lambda_k & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \Lambda_k \end{pmatrix}$$

Let us take

$$X_1 A_1 + X_2 A_2 + \cdots + X_m A_m = 0 \in M_{m,1}(P_n), \tag{2.1}$$

where $AX_i = X_i \Lambda_i$, X_i of full rank, $A_i \in P_n(\mathbb{C})$ $(i = 1, 2, \dots, m)$.

Equation (2.1) can be written as

$$IX_1 A_1 + IX_2 A_2 + \cdots + IX_m A_m = 0 \in M_{m,1}(P_n),$$

or, using the Kronecker product, as

$$(I_m \otimes I_n) X_1 A_1 + (I_m \otimes I_n) X_2 A_2 + \cdots + (I_m \otimes I_n) X_m A_m = 0 \in M_{m,1}(P_n). \tag{2.2}$$

Multiplying (2.1) on the left by A gives

$$AX_1A_1 + AX_2A_2 + \cdots + AX_mA_m = 0 \in M_1(P_n),$$

or

$$X_1\Lambda_1A_1 + X_2\Lambda_2A_2 + \cdots + X_m\Lambda_mA_m = 0 \in M_1(P_n),$$

and, again using the Kronecker product, this is equivalent to

$$(I_m \otimes \Lambda_1)X_1A_1 + (I_m \otimes \Lambda_2)X_2A_2 + \cdots + (I_m \otimes \Lambda_m)X_mA_m = 0 \in M_{m,1}(P_n). \tag{2.3}$$

A simple induction argument shows that

$$(I_m \otimes \Lambda_1^r)X_1A_1 + (I_m \otimes \Lambda_2^r)X_2A_2 + \cdots + (I_m \otimes \Lambda_m^r)X_mA_m = 0 \in M_{m,1}(P_n) \qquad (r = 0,1,\ldots,m-1). \tag{2.4}$$

In this way we have obtained a system of matrix equations—with unknown $X_1A_1, X_2A_2, \ldots, X_mA_m$—which we write as follows:

$$\begin{pmatrix} I_m \otimes I_n & I_m \otimes I_n & \cdots & I_m \otimes I_n \\ I_m \otimes \Lambda_1 & I_m \otimes \Lambda_2 & \cdots & I_m \otimes \Lambda_m \\ \cdots & \cdots & \cdots & \cdots \\ I_m \otimes \Lambda_1^{m-1} & I_m \otimes \Lambda_2^{m-1} & \cdots & I_m \otimes \Lambda_m^{m-1} \end{pmatrix} \begin{pmatrix} X_1A_1 \\ X_2A_2 \\ \vdots \\ X_mA_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in M_{m^2,1}(P_n). \tag{2.5}$$

From the system (2.5) one obtains $X_iA_i = 0 \in M_{m,1}(P_n)$ $(i = 1,2,\ldots,m)$ if and only if the coefficient matrix of (2.5)—which is an $m^2n \times m^2n$ complex matrix and belongs to $M_{m^2}(P_n)$—is nonsingular.

As X_i $(i=1,2,\ldots,n)$ is, by definition, of full rank, we have, denoting by X_i^L the left inverse of X_i,

$$\begin{aligned} X_iA_i=0 \quad (i=1,2,\ldots,m) \quad &\Leftrightarrow \quad X_i^LX_iA_i=0 \quad (i=1,2,\ldots,m) \\ &\Leftrightarrow \quad I_nA_i=0 \quad (i=1,2,\ldots,m) \\ &\Leftrightarrow \quad A_i=0 \quad (i=1,2,\ldots,m). \end{aligned}$$

From the above equivalence it follows that $A_i=0\in P_n(\mathbf{C})$ $(i=1,2,\ldots,m)$ if and only if the coefficient matrix of the system (2.5) is nonsingular.

Let us analyze the coefficient matrix of (2.5). By exchanging block rows and block columns, we obtain the scalar block product

$$V(\Lambda_1\Lambda_2\cdots\Lambda_m)I\in M_{m^2}(P_n). \tag{2.6}$$

We say that the value of the determinant of the coefficient matrix of the system (2.5) and the value of the determinant of the (block Vandermonde) matrix $V(\Lambda_1\Lambda_2\cdots\Lambda_m)$ are simultaneously equal to zero or simultaneously different from zero. Indeed by (1.2), we have, only to remark that

$$\det\begin{pmatrix} I_m\otimes I_n & I_m\otimes I_n & \cdots & I_m\otimes I_n \\ I_m\otimes\Lambda_1 & I_m\otimes\Lambda_2 & \cdots & I_m\otimes\Lambda_m \\ \cdots & \cdots & \cdots & \cdots \\ I_m\otimes\Lambda_1^{m-1} & I_m\otimes\Lambda_2^{m-1} & \cdots & I_m\otimes\Lambda_m^{m-1} \end{pmatrix}$$

$$=\left[\det\begin{pmatrix} I_n & I_n & \cdots & I_n \\ \Lambda_1 & \Lambda_2 & \cdots & \Lambda_m \\ \cdots & \cdots & \cdots & \cdots \\ \Lambda_1^{m-1} & \Lambda_2^{m-1} & \cdots & \Lambda_m^{m-1} \end{pmatrix}\right]^m$$

and the proof is achieved. ■

Now we are in a position to study the similarity of matrices partitioned into commuting blocks.

PROPOSITION 2.2. *Let* $\{\Lambda_1,\Lambda_2,\ldots,\Lambda_m\}$ *be a complete set of block eigenvalues* $\Lambda_i\in P_n(\mathbf{C})$ *of a matrix* $A\in M_m(P_n)$, *such that the block Vandermonde matrix* $V(\Lambda_1\Lambda_2\cdots\Lambda_m)$ *is nonsingular. Then the matrices* A

and $\text{diag}(\Lambda_1\Lambda_2\cdots\Lambda_m)$ *are block-similar, that is to say,*

$$(X_1\cdots X_m)^{-1}A(X_1\cdots X_m)=\text{diag}(\Lambda_1\cdots\Lambda_m)$$

where $AX_i=X_i\Lambda_i$, $X_i\in M_{m,1}(P_n)$ $(i=1,2,\dots,m)$, *and each* X_i *is of full rank.*

Proof. We have

$$AX_i=X_i\Lambda_i \quad (i=1,2,\dots,m) \quad \Leftrightarrow \quad A(X_1\cdots X_m)$$
$$=(X_1\cdots X_m)\text{diag}(\Lambda_1\cdots\Lambda_m);$$

but the block vectors $X_i\in M_{m,1}(P_n)$ $(i=1,2,\dots,m)$ are, by Proposition 2.1, block-linearly independent. Hence, by Lemma 2.1, the matrix $(X_1\cdots X_m)$ is nonsingular. This ends the proof. ∎

In the following result we show that block-similar matrices have the same block eigenvalues:

PROPOSITION 2.3. *Consider the relation*

$$(X_1\cdots X_m)^{-1}A(X_1\cdots X_m)=\text{diag}(\Lambda_1\cdots\Lambda_m), \tag{*}$$

where $A\in M_m(P_n)$, $\Lambda_i\in P_n(\mathbb{C})$, $X_i\in M_{m,1}(P_n)$ $(i=1,2,\dots,m)$. *Then:*

(1) *the block eigenvalues of* $\text{diag}(\Lambda_1\cdots\Lambda_m)\in M_m(P_n)$ *are block eigenvalues of* A;

(2) *the block eigenvalues of* A *are block eigenvalues of* $\text{diag}(\Lambda_1,\dots,\Lambda_m)$.

Proof. (1): Let us suppose that

$$\text{diag}(\Lambda_1\cdots\Lambda_m)B=B\Lambda, \tag{**}$$

where $B\in M_{m,1}(P_n)$ is of full rank and $\Lambda\in P_n(\mathbb{C})$, that is to say, Λ is a block eigenvalue of $\text{diag}(\Lambda_1\cdots\Lambda_m)$. Let us take $X=(X_1\cdots X_m)B$. We obtain, from (*),

$$AX=(X_1\cdots X_m)\text{diag}(\Lambda_1\cdots\Lambda_m)B,$$

and from (**),

$$AX = (X_1 \cdots X_m)\, B\Lambda = X\Lambda.$$

But, as B is of full rank and $(X_1 \cdots X_m)$ is nonsingular, $X \in M_{m,1}(P_n)$ is of full rank. Hence Λ is a block eigenvalue of A.

(2): We suppose that

$$AX = X\Lambda,$$

where $X \in M_{m,1}(P_n)$ is of full rank and $\Lambda \in P_n(\mathbb{C})$. Taking into account the relation (*), we get

$$\operatorname{diag}(\Lambda_1 \cdots \Lambda_m)\left[(X_1 \cdots X_m)^{-1} X\right] = \left[(X_1 \cdots X_m)^{-1} X\right]\Lambda.$$

Hence Λ is a block eigenvalue of $\operatorname{diag}(\Lambda_1 \cdots \Lambda_m)$, associated with the block eigenvector $(X_1 \cdots X_m)^{-1}X$. ■

REMARK 2.1. In Proposition 2.3, $X_i \in M_{m,1}(P_n)$ $(i = 1, 2, \ldots, m)$ are the block eigenvectors associated with the block eigenvalues $\Lambda_i \in P_n(\mathbb{C})$ $(i = 1, 2, \ldots, m)$, as is easily seen.

The results of Proposition 2.3, which applies when the blocks are not commutative, can be found in [1].

3. BLOCK COMPOUND MATRICES

In this section we shall introduce a generalization of compound matrices and examine the block eigenvalues of such a block compound matrix.

We construct the block-compound matrix of a given matrix in the same way as the usual compound [3, 6]. (We use the lexicographical order.)

DEFINITION 3.1. We pick a natural number p, and for every $m \geqslant p$ we enumerate lexicographically the $\binom{m}{p}$ subsets of $\{1, 2, \ldots, m\}$ with p elements. Then we assign to the matrix $A \in M_m(P_n)$ its block compound

$$D_p(A) \in M_N(P_n), \qquad N = \binom{m}{p},$$

where each block is the formal determinant of a submatrix of A having p block rows and p block columns.

We are going to define a relation between the block eigenvalues of A and the block eigenvalues of $D_p(A)$. The needed proposition is a generalization of a result referred to in [3,6] and is as follows:

PROPOSITION 3.1. *Let $\{\Lambda_1,\dots,\Lambda_m\}$ be a complete set of block eigenvalues $\Lambda_i \in P_n(\mathbb{C})$ of the matrix $A \in M_m(P_n)$ such that the block Vandermonde matrix $V(\Lambda_1 \cdots \Lambda_m)$ is nonsingular. Then the block eigenvalues of the p-block-compound $D_p(A)$ are obtained by considering all the $\binom{m}{p}$ products of $\Lambda_1,\dots,\Lambda_m$ with distinct subscripts.*

Proof. Let us suppose that we have

$$(X_1 \cdots X_m)^{-1} A (X_1 \cdots X_m) = \operatorname{diag}(\Lambda_1 \cdots \Lambda_m), \tag{3.1}$$

where $A \in M_m(P_n)$, $\Lambda_i \in P_n(\mathbb{C})$, $X_i \in M_{m,1}(P_n)$ $(i=1,2,\dots,m)$. The relation (3.1) is equivalent to

$$\chi^{-1} A \chi = \mathfrak{D}. \tag{3.2}$$

As $\chi \in M_m(P_n)$, hence $\chi^{-1} \in M_m(P_n)$ [5].

By use of the so-called Binet-Cauchy relations (applied to our context) and following an argument similar to that used in [3] in the scalar case, we have

$$D_p(\chi^{-1} A \chi) = D_p(\chi^{-1}) D_p(A) D_p(\chi); \tag{3.3}$$

hence

$$D_p(\chi^{-1}) D_p(A) D_p(\chi) = D_p(\mathfrak{D}). \tag{3.4}$$

We have also

$$D_p(\chi^{-1}) = [D_p(\chi)]^{-1}. \tag{3.5}$$

We obtain finally the relation

$$[D_p(\chi)]^{-1} D_p(A) D_p(\chi) = D_p(\mathfrak{D}), \tag{3.6}$$

or, in other words, the matrices $D_p(A)$ and $D_p(\mathcal{D})$ are block-similar. We can write (3.6) in the following way:

$$(Y_1 \cdots Y_N)^{-1} D_p(A)(Y_1 \cdots Y_N) = \operatorname{diag}(U_1 \cdots U_N) \tag{3.7}$$

with $N = \binom{m}{p}$. As, by Proposition 2.3, the matrices $D_p(A)$ and $\operatorname{diag}(U_1 \cdots U_N)$ have the same block eigenvalues, the proof is achieved. ■

4. ADDITIVE BLOCK COMPOUND MATRICES

Here we generalize the concept of additive compound matrix.

DEFINITION 4.1. The p-additive-block-compound matrix of $A \in M_m(P_n)$ is the matrix

$$\Delta_p(A) \in M_N(P_n), \qquad N = \binom{m}{p},$$

that appears as the coefficient of the variable $t \in \mathbb{C}$ in $D_p(I_{mn} + tA)$.

We are looking for a relation between the block eigenvalues of a matrix A and the block eigenvalues of the matrix $\Delta_p(A)$. The needed proposition is a generalization of a result mentioned in [2, 6] and is as follows:

PROPOSITION 4.1. *Let $\{\Lambda_1, \dots, \Lambda_m\}$ be a complete set of block eigenvalues $\Lambda_i \in P_n(\mathbb{C})$ of the matrix $A \in M_m(P_n)$ such that the block-Vandermonde matrix $V(\Lambda_1 \cdots \Lambda_m)$ is nonsingular. Then the block eigenvalues of the p-additive-block-compound $\Delta_p(A)$ are obtained by considering all the $\binom{m}{p}$ sums of $\Lambda_1, \dots, \Lambda_m$ with distinct subscripts.*

Proof. Let us suppose that we have the relation

$$(X_1 \cdots X_m)^{-1} A (X_1 \cdots X_m) = \operatorname{diag}(\Lambda_1 \cdots \Lambda_m), \tag{4.1}$$

where $A \in M_m(P_n)$, $X_i \in M_{m,1}(P_n)$, $\Lambda_i \in P_n(\mathbb{C})$ $(i = 1, 2, \dots, m)$. This is equivalent to

$$\chi^{-1} A \chi = \mathcal{D}. \tag{4.2}$$

As seen in Section 3, we have

$$\left[D_p(X_1\cdots X_m)\right]^{-1}D_p(A)D_p(X_1\cdots X_m)=D_p(\mathfrak{D}). \tag{4.3}$$

But, remarking that A and $I_{mn}+tA$ have the same block eigenvectors, we can write

$$\left[D_p(X_1\cdots X_m)\right]^{-1}D_p(I_{mn}+tA)D_p(X_1\cdots X_m)=D_p(I_{mn}+t\mathfrak{D}). \tag{4.4}$$

We can write (4.4) in the following manner:

$$(Y_1\cdots Y_N)^{-1}D_p(I_{mn}+tA)(Y_1\cdots Y_N)=D_p(I_{mn}+t\mathfrak{D}),\qquad N=\binom{m}{p}. \tag{4.5}$$

From (4.5) we have successively

$$(Y_1\cdots Y_N)^{-1}\left[I_{Nn}+t\Delta_p(A)+t^2(\cdots)\right](Y_1\cdots Y_N)=I_{Nn}+t\Delta_p(\mathfrak{D}) \tag{4.6}$$

if and only if

$$\begin{aligned}(Y_1\cdots Y_N)^{-1}I_{Nn}(Y_1\cdots Y_N)+t(Y_1\cdots Y_N)^{-1}\Delta_p(A)(Y_1\cdots Y_N)\\ +t^2(Y_1\cdots Y_N)^{-1}(\cdots)(Y_1\cdots Y_N)=I_{Nn}+t\Delta_p(\mathfrak{D})+t^2(\cdots)\end{aligned} \tag{4.7}$$

if and only if

$$\begin{aligned}I_{Nn}+t(Y_1\cdots Y_N)^{-1}\Delta_p(A)(Y_1\cdots Y_N)+t^2(Y_1\cdots Y_N)^{-1}(\cdots)(Y_1\cdots Y_N)\\ =I_{Nn}+t\Delta_p(\mathfrak{D})+t^2(\cdots).\end{aligned} \tag{4.8}$$

By comparison of the two sides of this relation, we get

$$(Y_1\cdots Y_N)^{-1}\Delta_p(A)(Y_1\cdots Y_N)=\Delta_p(\mathfrak{D}),\qquad N=\binom{m}{p}, \tag{4.9}$$

or, in other words, the matrices $\Delta_p(A)$ and $\Delta_p(\mathcal{D})$ are block-similar. Then, from (4.9) and by Proposition 2.3, the matrices $\Delta_p(A)$ and $\Delta_p(\mathcal{D})$ have the same block eigenvalues.

At this stage we are going to study the block eigenvalues of $\Delta_p(\mathcal{D})$. We have

$$D_p(I_{mn}+t\mathcal{D}) = \begin{pmatrix} (I_n+t\Lambda_{i_1})\cdots(I_n+t\Lambda_{i_p}) & & \bigcirc \\ & * & \\ \bigcirc & & \ddots \\ & & & * \end{pmatrix}, \tag{4.10}$$

a block-diagonal matrix where the diagonal blocks are the $\binom{m}{p}$ products of p factors of the form $(I_n + t\Lambda_{k_i})$ with distinct subscripts. Hence we obtain

$$D_p(I_{mn}+t\mathcal{D}) = \begin{pmatrix} I_n & & \\ & \ddots & \\ & & I_n \end{pmatrix}$$

$$+t\begin{pmatrix} \Lambda_{i_1}+\cdots+\Lambda_{i_p} & & \bigcirc \\ & * & \\ \bigcirc & & \ddots \\ & & & * \end{pmatrix} + t^2(\cdots), \tag{4.11}$$

from which we conclude that matrix $\Delta_p(\mathcal{D})$ has diagonal blocks of the form $\Lambda_{i_1}+\Lambda_{i_2}+\cdots+\Lambda_{i_p}$.

The proof is then achieved by following an argument similar to the one employed in Section 3. ■

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